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Analytical results for the Sznajd model of opinion formation

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Abstract. The Sznajd model, which describes opinion formation and social influence, is treated analytically on a complete graph. We prove the existence of the phase transition in the original formulation of the model, while for the Ochrombel modification we find smooth behaviour without transition. We calculate the average time to reach the stationary state as well as the exponential tail of its probability distribution. An analytical argument for the observed $1/n$ dependence in the distribution of votes in Brazilian elections is provided.

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1 Introduction

There is significant convergence between statistical physics and mathematical sociology in approaches to their respective fields [1]. Ising model, the single most studied statistical physics model, finds its numerous applications in sociophysics simulations. Conversely, sociologically inspired models pose new challenges to statistical physics. We believe this is the case of the Sznajd model we are studying here.

The model of Sznajd-Weron and Sznajd [2] was designed to explain certain features of opinion dynamics. The slogan "United we stand, divided we fall" lead to simple dynamics, in which individuals placed on a lattice (one-dimensional in the first version) can choose between two opinions (political parties, products etc.) and in each update step a pair of neighbours sharing common opinion persuade their neighbours to join their opinion. Therefore, it was noted that contrary to the Ising or voter [3] models, information does not flow from the neighbourhood to the selected spin, but conversely, it flows out from the selected cluster to its neighbours.

The model initiated a surge of immediate interest [4–25] and the results of numerical simulations can be briefly summarised as follows. The results do not depend much on the spatial dimensionality or on the type of the neighbourhood selected [11]. In the case of q choices of opinion, the system has q obvious homogeneous stationary (absorbing) states, where all individuals choose the same opinion. There is no way to go out of the homogeneous state, so it is an attractor of the dynamics. This is reminiscent of a zero-temperature dynamics, which in Ising model leads to rich behaviour [26]. However, in the Sznajd model, the possible metastable states, like the "antiferromagnetic" configuration have negligible probability to occur, unless we introduce explicitly also an "antiferromagnetic" dynamic rule as it was used in the very first formulation [2].

The case $q = 2$ was studied mostly, denoting the opinions by Ising variables $+1$ and -1. The probability of hitting the stationary state of all $+1$ (or, complementary, all -1) was studied, depending on the initial fraction p of the individuals choosing $+1$. Sharp transition was observed at value $p = 0.5$ [11]; for $p > 0.5$ the probability to reach eventually the state of all opinions $+1$ is close to one, while for $p < 0.5$ it is negligible, which can be interpreted as a dynamical phase transition. The distribution of times needed to reach the stationary state was measured, revealing a peak followed by relatively fast decay. This means that the average hitting time is a well-defined quantity [11].

It was also found in one and two-dimensional lattices that the fraction of individuals who never changed opinion decays as a power with time, similarly to Ising model. While the exponent in one dimension agrees with the Ising case, the two-dimensional Sznajd model gives different exponent than Ising model, indicating different dynamical universality class [13]. Also the waiting time between two subsequent opinion changes is distributed according to a power-law [2].

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Among other studies, let us mention the influence of advertising effects [18,19] and price formation [20]. Long-range interactions were studied in [21].

In a very short but intriguing note [22] Ochrombel suggested a drastic simplification of the Sznajd model. In the Ochrombel version it is not necessary to have a cluster of identical opinions. Any individual is capable to convince her neighbours to select the same opinion. This model was reported to share all essential features of the original Sznajd model, only the phase transition in the probability of hitting the state of all $+1$ at $p = 0.5$ is absent.

The Sznajd model was also used to model the election process. There is recent empirical evidence from Brazilian elections [27–29] that the distribution of votes per candidate follows a power-law, more specifically $P(n) \sim 1/n$, where n is the number of votes. This result was reproduced in a study [4] based on Sznajd model on a scale-free network [30–32].

The dynamics of elections was thoroughly investigated by Galam [33–36], showing that majority rule applied on sufficiently many hierarchical levels leads to a homogeneous "totalitarian" state with one opinion pervading the whole system.

Other approaches to physical modelling of opinion dynamics were also investigated [37,38] and among them especially the Axelrod model, which was found to have rich behaviour from the statistical physics point of view [39–41].

We should also mention the well studied voter model [3,42–44], which is very similar in spirit to the Sznajd model. Indeed, the relation of the two models was studied *e.g.* in [45] and it seems that Sznajd model reduces to the voter model at least for certain setups (especially using the Ochrombel simplification on a complete graph) while for others the voter model can be generalised so that it includes the rules of Sznajd model as a special case. In fact, similar analysis to that presented here was performed for voter model, contact process and related processes in [46]. The persistence properties of the voter model on complete graph were studied in [42].

Very recently a "Majority rule" model, sharing some features with Sznajd model, was introduced and studied in [47] and its generalisation to the Majority-Minority model [48] gives in the mean-field approximation results closely related to ours.

2 Formulation of the model and its simplifications

2.1 General scheme

In the original formulation of the Sznajd model, the "united we stand" principle is often stressed [2,11]. It means that only a cluster of identical opinions can spread the same opinion toward its neighbours. However, this principle was relaxed in the Ochrombel simplification [22] without qualitatively affecting many of the results (except the presence of the phase transition). We will propose some other simplifications here, supposing the results remain robust.

Let us have N agents, each of which can be in one of q states (opinions) $\sigma \in S$. We may for example think of a q-state Potts model variables. Each agent sits on a node of a social network, and they can interact along the edges with their nearest neighbours.

The opinion of the agent i is denoted σ_i. The state of the system is described by the set of opinions of all the agents, $\Sigma = [\sigma_1, \sigma_2, ..., \sigma_N]$.

The variable $\Sigma(t)$ performs a discrete-time Markov process, whose transition probabilities from time t to $t+1$ differ in various cases, which will be specified in the following.

2.2 Case I: two against one

The first case investigated, which we will sometimes call "two against one", generalises and simultaneously simplifies the various versions introduced in [11]. The main difference is in the fact that we will change at maximum *one* agent at each time step. This may not significantly change the behaviour, as the various choices of neighbourhood in [11] exhibit only little difference.

Our algorithm will iterate the following three steps. First, choose randomly an agent i. Then, choose randomly one of its neighbours, say j. If $\sigma_i(t) \neq \sigma_j(t)$, nothing happens. However, if $\sigma_i(t) = \sigma_j(t)$, we will choose randomly one of the common neighbours of both i and j, say k, and set $\sigma_k(t+1) = \sigma_i(t)$. We may also write it schematically as reactions AAB → AAA, BBA → BBB.

2.3 Case II: Ochrombel simplification

In this case, we do not need to have two neighbours in the same state. Everybody can influence each of its neighbours. We choose an agent i at random. Then, choose j randomly among neighbours and set $\sigma_j(t+1) = \sigma_i(t)$. Therefore, the process may be written as AB → AA, BA → BB. In fact, on fully connected network the Ochrombel simplification is equivalent to voter model, whose dynamical properties were studied *e.g.* in [42].

As an obvious observation we can note that both in case I and case II the uniform states, with all σ_i equal, are stable under the dynamics. However, we can expect variety of metastable states in the case I, in which there are no pairs of neighbours in the same state, therefore the dynamics does not proceed any further.

3 On a fully-connected network

We will approximate the complex social network by the fully-connected network (the complete graph) of N nodes. Here, any two agents are neighbours; in the case I we simply choose three agents i, j, k at random and in the case II two agents i, j at random. Note that the order in which they are chosen matters. This makes our process different *e.g.* from the majority [47] or majority-minority [48] models, although on fully connected network the difference may consist only in rescaling certain variables.

We will call this setup a mean-field approximation in the same sense as the Ising model on the complete graph can be considered as an approximation for Ising model on hypercubic lattice of high dimensionality. Of course this is not a good approximation to the original one-dimensional formulation of the Sznajd model [2], but we believe it is appropriate for much more realistic studies of Sznajd model on complex networks [4,16,23]. We refer the reader to Appendix A for a more formal definition of the Sznajd model on an arbitrary graph.

In fully-connected network the state of the system is fully described by the occupation numbers $N_\sigma = \sum_{i=1}^{N} \delta_{\sigma_i \sigma}$, or equivalently the densities $n_\sigma = N_\sigma/N$, for each opinion $\sigma \in S$. The dynamics of these occupation numbers fully describes the evolution of the system. As the total number of nodes is conserved, there are $q-1$ independent dynamical variables.

Let us start with the case II (Ochrombel simplification) with only two opinions, $q = 2$. The variable σ can assume only two values, denoted $\sigma = \pm 1$ for convenience. Indeed, we are effectively working with Ising spins. The state is described by one dynamical variable only, which will be taken as a "magnetisation",

$$m = \frac{N_+ - N_-}{N}. \tag{1}$$

In one step of the dynamics, three events can happen. The magnetisation may remain constant or it can change by $\pm 2/N$. The probabilities of these three events can be easily calculated

$$\begin{aligned}
\text{Prob}\left\{m \to m + \frac{2}{N}\right\} =& \frac{1}{4}\left(1-m^2\right)\left(1 + \frac{1}{N-1}\right) \\
\text{Prob}\left\{m \to m - \frac{2}{N}\right\} =& \frac{1}{4}\left(1-m^2\right)\left(1 + \frac{1}{N-1}\right) \\
\text{Prob}\left\{m \to m\right\} =& \left(\frac{1}{2}\left(1+m^2\right) - \frac{1}{N}\right) \\
& \times \left(1 + \frac{1}{N-1}\right).
\end{aligned} \tag{2}$$

Our objective is writing the master equation for the probability density of the random variable $m(t)$, which we denote P_m. It can be found easily in the thermodynamic limit $N \to \infty$. Indeed, we find that the time should be rescaled as

$$t = N^2 \tau \tag{3}$$

in the thermodynamic limit. Then the probability density evolves according to the partial differential equation

$$\frac{\partial}{\partial \tau} P_m(m,\tau) = \frac{\partial^2}{\partial m^2}\left[\left(1-m^2\right) P_m(m,\tau)\right]. \tag{4}$$

The latter equation describes in principle fully the evolution of the Sznajd model in Ochrombel simplification on a complete graph. It has the form of a diffusion equation with position-dependent diffusion constant.

Let us turn now to the case I (original Sznajd model), again with $q = 2$. We may repeat step by step the considerations made above for the case II. Namely, our dynamical variable will be again the magnetisation m which may either remain unchanged or change by $\pm 2/N$ in one step. For the probabilities of these events we can find formulae analogous to (2)

$$\begin{aligned}
\text{Prob}\left\{m \to m + \frac{2}{N}\right\} &= \frac{\left(1-m^2\right)}{8}\left(1 + m + \frac{1+3\,m}{N}\right) \\
\text{Prob}\left\{m \to m - \frac{2}{N}\right\} &= \frac{\left(1-m^2\right)}{8}\left(1 - m + \frac{1-3\,m}{N}\right) \\
\text{Prob}\left\{m \to m\right\} &= 1 - \frac{\left(1-m^2\right)}{4}\left(1 + \frac{1}{N}\right)
\end{aligned} \tag{5}$$

where the terms of order $1/N^2$ are neglected. Note that the probabilities of changes $\pm 2/N$ are not symmetric, contrary to the previous case (II). This fact has all-important consequences. We will see later that it is responsible for the fact that the original Sznajd model exhibits phase transition, while in Ochrombel simplification the transition is absent.

A more immediate consequence is that the time must be rescaled differently, in order to get sensible thermodynamic limit, namely

$$t = 2N\,\tau. \tag{6}$$

The second consequence is that the equation for $P_m(m,\tau)$ contains *first* derivative with respect to m, representing a pure drift in magnetisation:

$$\frac{\partial}{\partial \tau} P_m(m,\tau) = -\frac{\partial}{\partial m}\left[(1-m^2) m\, P_m(m,\tau)\right]. \tag{7}$$

Contrary to the previous case (4) the diffusion term, containing the second derivative in m, represents only the finite-size correction to the drift term. However, this correction may dominate close to points $m = \pm 1$ and $m = 0$ where the drift velocity becomes zero.

Next case investigated will be the case II with arbitrary value of q. Moreover, we will assume that the number of opinions is large, $q \gg 1$. Let us define the distribution of occupation numbers

$$D(n) = \frac{N}{q}\sum_{\sigma=1}^{q} \delta\left(n - n_\sigma\right) \tag{8}$$

where $\delta(x) = 1$ for $x = 0$ and zero elsewhere. It would be much more difficult to write the full dynamic equation for $D(n)$. Therefore, we use the approximation which replaces the distribution $D(n)$ by its configuration average $P_n(n) = \langle D(n)\rangle$. In the limit $N \to \infty$ and $q \to \infty$ and substituting the variable $x = 2n - 1$ we arrive at the equation

$$\frac{\partial}{\partial \tau} P_n(x,\tau) = \frac{\partial^2}{\partial x^2}\left[\left(1-x^2\right) P_n(x,\tau)\right]. \tag{9}$$

The time is rescaled again according to the equation (3). We can see that the equations (4) and (9) have identical form, although the interpretation of variables is different. We can therefore solve the two cases simultaneously. This will be performed in the next section.

4 Solution of the dynamics

4.1 Two against one: case I

The case I, $q = 2$ is described by the equation

$$\frac{\partial}{\partial \tau} P(x,\tau) = -\frac{\partial}{\partial x}\left[(1-x^2)x\, P(x,\tau)\right]. \quad (10)$$

It can be easily verified that the solution has the following general form

$$P(x,\tau) = [(1-x^2)x]^{-1} f\left(\mathrm{e}^{-\tau}\frac{x}{\sqrt{1-x^2}}\right) \quad (11)$$

for arbitrary function $f(y)$. The form of the function $f(y)$ is given by initial conditions. For example if the initial condition is a δ-function, it keeps the same form during the evolution, only the location shifts in time. This way we could in principle calculate, how long it takes to reach the edges of the interval from given initial position. This would be the time to reach the stationary state. However, it comes out that the time needed blows up. The reason comes from the infinite-size limit $N \to \infty$. Indeed, very close to the points $x = \pm 1$ the finite-size effects take over.

We can estimate the average time needed to reach the stationary state in finite system by the following consideration. In fact, the equation (10) describes the drift which pushes the system toward the stationary state, but neglects the effect of diffusion, which becomes important at a distance $\sim 1/N$ from the points $x = \pm 1$. Therefore, we must calculate the time necessary for the drift to drive the system to the point $\pm(1 - 1/N)$. The initial fraction p of opinions $+1$ corresponds to the initial condition $x_0 = 2p - 1$ and from the formula (11) we have the following estimate for the average time $\langle \tau_{\mathrm{st}} \rangle$ to reach the stationary state

$$\langle \tau_{\mathrm{st}} \rangle \simeq -\ln\left(\frac{|2p-1|}{\sqrt{p(1-p)}}\frac{1}{\sqrt{N}}\right). \quad (12)$$

It is also possible to include the correction terms of order $O(1/N)$ into equation (10) and deduce the equation for the average time to reach the absorbing state $\langle \tau_{\mathrm{st}} \rangle(x_0)$ on condition that the process started at initial position x_0. Following the general scheme [49] we obtain a second-order ordinary differential equation

$$\left(1+\frac{3}{N}\right)\left(1-{x_0}^2\right) x_0 \frac{\mathrm{d}}{\mathrm{d}x_0}\langle \tau_{\mathrm{st}} \rangle (x_0) + \frac{1}{N}\left(1-{x_0}^2\right)\frac{\mathrm{d}^2}{\mathrm{d}{x_0}^2}\langle \tau_{\mathrm{st}} \rangle (x_0) = -1. \quad (13)$$

The solution of (13) is

$$\langle \tau_{\mathrm{st}} \rangle (x_0) = N \int_{-1}^{x_0}\int_{y}^{0} \frac{\mathrm{e}^{\frac{N+3}{2}z^2}}{1-z^2}\mathrm{d}z\, \mathrm{e}^{-\frac{N+3}{2}y^2}\,\mathrm{d}y. \quad (14)$$

Indeed, for x_0 not too close to either of the points $x_0 = -1, 0, 1$ (the distance must be large compared to $1/N$) we obtain from the formula (14) an approximate expression of the form given in (12). Another way to obtain the same p dependence as in (12) is to omit the $O(1/N)$ terms in the equation (13) and solve the first-order differential equation. In this case, however, we lose any information about the dependence on N. We should also note that a result essentially equivalent to equation (12) was obtained also in [47].

It is rather interesting to observe that the deterministic dynamics of Galam model [34,36] leads to a formula very similar to (12), while the interpretation of the time variable is totally different: in Galam model it represents the number of hierarchical levels on which the majority rule is iterated.

It would be desirable to calculate the full probability distribution for the time to reach the stationary state τ_{st} and not only the average. That is possible using again the formalism of adjoint equation [49], when we introduce the $1/N$ corrections to equation (10) but the resulting partial differential equation is difficult to solve explicitly. Instead, we estimate the exponential tail of the distribution by a simple consideration.

Indeed, after the drift had pushed the system to the state in which there is only single spin -1 immersed in a sea of all $+1$-s it finally comes into uniform stationary state if the first pair of spins chosen is both $+1$ and the third one is the single -1. This choice has probability $\simeq 1/N$. Therefore, the relaxation time toward the uniform state is $t_{\mathrm{relax}} \simeq N$ and using the scaling (6) we have for the tail of the distribution

$$P(\tau_{\mathrm{st}}) \sim \exp\left(-\frac{\tau_{\mathrm{st}}}{\tau_{\mathrm{relax}}}\right), \quad \tau_{\mathrm{st}} \to \infty \quad (15)$$

with

$$\tau_{\mathrm{relax}} \simeq \frac{1}{2}. \quad (16)$$

The most important observation we can draw from the solution (11) is the presence of the dynamic phase transition, as observed in numerical simulations. Indeed, starting with any fixed positive magnetisation, we have initial condition $P(x,0) = \delta(x - x_0)$, $x_0 > 0$, and the drift expressed by equation (11) always take us to the state with all agents having opinion $+1$, while from any state with negative magnetisation the drift leads the system eventually to the state with all agents having opinion -1 and the probability of ending in the state of all $+1$ is therefore $P_+ = \theta(p - 1/2)$. The possible deviations from this rule close to the zero magnetisation (*i.e.* $p = 0.5$) are due to the finite size effects, which are neglected in (10). The presence of the phase transition is also indicated by the divergence of the average time to reach the stationary state (12) for $p \to 1/2$.

4.2 Ochrombel simplification: case II

The equation

$$\frac{\partial}{\partial \tau} P(x,\tau) = \frac{\partial^2}{\partial x^2}\left[(1-x^2)\, P(x,\tau)\right] \tag{17}$$

describes both the case II, $q=2$ and II, $q \gg 1$, only the interpretation of the variable x differ: in the former case it corresponds to the magnetisation, while in the latter case it is shifted percentage of votes. By solving equation (17) we treat simultaneously both cases.

The equation of the form (17) was already studied in variety of contexts, *e.g.* population genetics [50,51] or reaction kinetics [52] and can be tackled by standard methods developed for Fokker-Planck equation.

Indeed, we look for the solution using the expansion in eigenvectors. We can write (17) it in the form $\frac{\partial}{\partial\tau}P(x,\tau) = \mathcal{L}P(x,\tau)$ where the linear operator $\mathcal{L}$ acts as $(\mathcal{L}f)(x) = \frac{\partial^2}{\partial x^2}\left[(1-x^2)\, f(x)\right]$. We therefore need to find the set of eigenvectors of $\mathcal{L}$. Denoting $\Phi_c(x)$ the eigenvector corresponding to the eigenvalue $-c$, we have the following equation

$$(1-x^2)\, \Phi_c''(x) - 4x\, \Phi_c'(x) + (c-2)\, \Phi_c(x) = 0. \tag{18}$$

The full solution of (17) can be then expanded as

$$P(x,\tau) = \sum_c A_c \mathrm{e}^{-c\tau}\, \Phi_c(x) \tag{19}$$

with coefficients A_c determined from the initial condition.

Important question to be settled prior to the attempt for solution is, what is the appropriate space of functions $\Phi(x)$. First, the interpretation of these functions as probability densities sets the requirement that it must be normalisable: $\int \Phi(x)\, \mathrm{d}x < \infty$. Second, only the interval $x \in [-1,1]$ is relevant, so $\Phi(x) = 0$ outside this interval. Finally, we should anticipate the possibility that δ-functions appear in the solution, namely located at $x = \pm 1$, because the uniform states, with all sites carrying the same spin value, are stable under the dynamics.

We therefore look for the solution of (18) in the space of distributions (*i.e.* linear functionals on sufficiently differentiable functions) with support restricted to the interval $[-1,1]$.

It is straightforward to find the eigenvectors corresponding to eigenvalue $c=0$, *i.e.* the stationary solutions of equation (17). They are composed of δ-functions only. In fact, the corresponding eigensubspace is two-dimensional and the base vectors can be chosen as

$$\Phi_{01} = \delta(x-1), \quad \Phi_{02} = \delta(x+1). \tag{20}$$

For $c \neq 0$ we first decompose the solution in ordinary function of x plus a pair of δ-functions, namely

$$\Phi_c = \phi_{c+}\, \delta(x-1) + \phi_{c-}\, \delta(x+1) + \phi_c(x)\, \theta(x-1)\, \theta(x+1) \tag{21}$$

where ϕ_{c+} and ϕ_{c-} are real numbers and $\phi_c(x)$ is a real doubly differentiable function. Then, equation (18) translates into equation for $\phi_c(x)$

$$(1-x^2)\, \phi_c''(x) - 4x\, \phi_c'(x) + (c-2)\, \phi_c(x) = 0 \tag{22}$$

accompanied by two other conditions

$$\lim_{x\to\pm1} \phi_c(x) = -\frac{c}{2}\, \phi_{c\pm}. \tag{23}$$

The general solution of equation (22) exhibits behaviour $\phi_c(x) \sim (1 \mp x)^\alpha$ at $x \to \pm 1$, where either $\alpha = 0$ or $\alpha = -1$. However, the latter case should be excluded, as it gives non-normalisable probability distribution. In fact it is the condition of normalisability that determines all possible eigenvalues c. The solution of (22) with correct behaviour at $x \to \pm 1$ can be expressed in Gegenbauer polynomials [52–54]. The eigenvalues are $c = c_l \equiv (l+1)(l+2)$ for $l = 0, 1, 2, ...$ An elementary solution and the table of several lowest polynomials is presented in Appendix B.

It is important to note that for any eigenvalue $c > 0$ we have

$$\int \Phi_c(x)\, \mathrm{d}x = 0 \qquad \int x\, \Phi_c(x)\, \mathrm{d}x = 0. \tag{24}$$

The consequence is that both $\int P(x,\tau)\mathrm{d}x$ and $\int xP(x,\tau)\mathrm{d}x$ are independent of time. While the first conservation law expresses simply the conservation of probability, the second one is a non-trivial consequence of the model dynamics. Mathematically it is related to the fact that the eigenspace corresponding to zero eigenvalue is two-dimensional.

Thus, we found the set of *right* eigenvectors of the operator $\mathcal{L}$. For practical solution we still need to establish the coefficients A_c in equation (19). To this end we need also the set of *left* eigenvectors of $\mathcal{L}$, checking simultaneously that the set of left and right eigenvalues coincide. First, we need to establish the adjoint operator to $\mathcal{L}$, defined by usual relation $(\mathcal{L}f|g) = (f|\mathcal{L}^T g)$. While $\mathcal{L}$ acts on the space of distributions, its adjoint $\mathcal{L}^T$ acts on the corresponding dual space, which is the space of sufficiently differentiable functions. Straightforward algebra gives $(\mathcal{L}^T g)(x) = (1-x^2)\, g''(x)$ which implies the following equation for the left eigenvectors

$$(1-x^2)\, \psi_c''(x) + c\, \psi_c(x) = 0. \tag{25}$$

We find again that for $c=0$ the eigensubspace is two-dimensional. We can choose the basis vectors so that they are mutually ortho-normal to the pair of right eigenvectors (20), namely

$$\psi_{01} = \frac{1}{2}(1+x), \quad \psi_{02} = \frac{1}{2}(1-x). \tag{26}$$

The solutions of (25) for $c > 0$ with proper boundary conditions are again polynomials presented in more detail in Appendix B.

The coefficients in the solution (19) with initial condition $P(x,0) = P_0(x)$ are then calculated as

$$A_c = \frac{\int P_0(x)\psi_c(x)\, \mathrm{d}x}{\int \phi_c(x)\psi_c(x)\, \mathrm{d}x}. \tag{27}$$

From the solution (19) we can deduce an important feature for the distribution of waiting times needed to reach the stationary state. Indeed, if $P_{\mathrm{st}}(\tau)$ is the probability density for ending at time τ in the stationary frozen configuration with all agents in the same state, we can express the probability that the stationary configuration was not reached before time τ as

$$P_{\mathrm{st}}^{>}(\tau) \equiv \int_{\tau}^{\infty} P_{\mathrm{st}}(\tau')\mathrm{d}\tau' \\ = 1 - \lim_{\epsilon \to 0^+} \left(\int_{-1-\epsilon}^{-1+\epsilon} + \int_{1-\epsilon}^{1+\epsilon} \right) P(x,\tau)\mathrm{d}x. \quad (28)$$

We can see that only the δ-function components of the eigenvectors $\Phi_c(x)$ in the expansion (19) contribute to $P_{\mathrm{st}}^{>}(\tau_{\mathrm{st}})$. More explicitly, we find

$$P_{\mathrm{st}}^{>}(\tau) = \sum_{c>0} 2A_c \, \frac{\phi_c(-1) + \phi_c(1)}{c} \, \mathrm{e}^{-c\tau}. \quad (29)$$

As the spectrum of eigenvalues is discrete, for long times only the lowest non-zero c (equal to $c_0 = 2$) is relevant. Therefore, the distribution of waiting times will have an exponential tail $P_{\mathrm{st}}^{>}(\tau) \sim \mathrm{e}^{-2\tau}, \ \tau \to \infty$. For initial condition $P_0(x) = \delta(x - x_0)$ we can easily compute also the prefactor in the leading term for large τ. Indeed, from (27) we get A_2 and finally obtain

$$P_{\mathrm{st}}^{>}(\tau) \simeq \frac{6}{4}(1 - x_0^2)\,\mathrm{e}^{-2\tau}, \ \tau \to \infty. \quad (30)$$

As the functions $\phi_c(x)$ are odd for $c = c_l$ with odd l, we should expect that the corrections to the formula (30) will be governed by the second next eigenvalue $c_2 = 12$. We will see later how it can be checked in numerical simulations.

As in the case I the average time $\langle \tau_{\mathrm{st}} \rangle (x_0)$ to reach the absorbing state when starting at position x_0 can be obtained, using the general formalism [49], from the equation

$$\left(1 - {x_0}^2\right) \, \frac{\mathrm{d}^2}{\mathrm{d}{x_0}^2} \langle \tau_{\mathrm{st}} \rangle (x_0) = -1 \quad (31)$$

which can be solved easily

$$\langle \tau_{\mathrm{st}} \rangle (x_0) = -\frac{x_0}{2} \ln \frac{1 + x_0}{1 - x_0} - \frac{1}{2} \ln \frac{1 - {x_0}^2}{4} \quad (32)$$

(see also [52,53]). The method of adjoint equation [49,53] can be used to calculate the distribution of times to reach the absorbing state, when starting from initial position at $x = x_0$, yielding results equivalent to our direct calculation. Indeed, inserting the initial condition $P_0(x) = \delta(x - x_0)$ into (27) we can see that the expression (29) represents an expansion in the eigenvectors $\psi_c(x_0)$ of the adjoint operator $\mathcal{L}^T$ taken at point x_0.

Contrary to the case I, we do not observe any phase transition here. This is due to the conservation of average magnetisation in the dynamics [47]. From this fact it follows immediately that $P_+ = p$. This result can be confirmed by an explicit calculation. Starting with fixed magnetisation $x_0 = 2p - 1$, the initial condition $P(x, 0) = \delta(x - x_0)$ broadens under the diffusive dynamics (17) and leaves always non-zero probability of ending in either of the possible stationary states. We already noted that $\int x\, P(x,\tau)\, \mathrm{d}x$ is independent of time under the dynamics (17). Therefore, the asymptotic state is the following combination of the eigenvectors (20) with $c = 0$

$$\lim_{\tau \to \infty} P(x,\tau) = \frac{1 - x_0}{2}\, \delta(x + 1) + \frac{1 + x_0}{2}\, \delta(x - 1) \quad (33)$$

and the probability of ending in the state of all +1 is therefore simply $P_+ = p$.

4.3 Distribution of votes

As already stressed in Section 3, equation (17) describes also the evolution of the distribution of votes in the case of $q \gg 1$ parties. We will present an argument how our results may explain the empirical data, suggesting the $1/n$ law for the distribution of votes.

As stressed in the discussion following equation (23), the time-independent solutions of equation (17) can behave either as $1 + x$ or $(1 + x)^{-1}$ in the limit $x \to -1$. However, the latter case was excluded by the requirement of normalisability of the probability density. On the other hand, relaxing the normalisability condition, the functions

$$\tilde{\phi}_{01}(x) = \frac{1}{1 + x} \quad (34)$$

$$\tilde{\phi}_{02}(x) = \frac{1}{1 - x} \quad (35)$$

are solutions of (22) with eigenvalue $c = 0$. (Of course, any linear combination of them is also solution with $c = 0$.)

How should be any of these additional solutions interpreted? The zero eigenvalue suggest that the function is stationary in time. However, it is not normalisable, therefore this solution cannot be reached from any initial condition. But if the distribution $P_n(x,\tau)$ is close to $\tilde{\phi}_{01}(x)$ (or $\tilde{\phi}_{02}(x)$) in some interval I of x, it is probable that it $P_n(x,\tau)$ will remain close to (34) (or (35), respectively) for certain period of time, while the interval I will gradually shrink and eventually disappear. Therefore, we may suggest (34) and (35) as a metastable states, or long-lived transient states.

This may explain the observation from simulations performed in [4]. In this work, the distribution of the type $1/n$ is obtained in a suitably chosen transient regime, in certain range of n. As $x = 2n - 1$, the behaviour of (34) at $x \to -1$ corresponds precisely to $1/n$ behaviour for small n.

A slightly more rigorous variant of the above argument is also possible. Imagine now that the political system represented by the set of opinions S is not closed, but new opinions may appear, replacing other ones which vanish.

Indeed, the current induced by the dynamics of case II can be read off from equation (17)

$$j = -\frac{\partial}{\partial x}[(1 - x^2)P(x,\tau)] \quad (36)$$

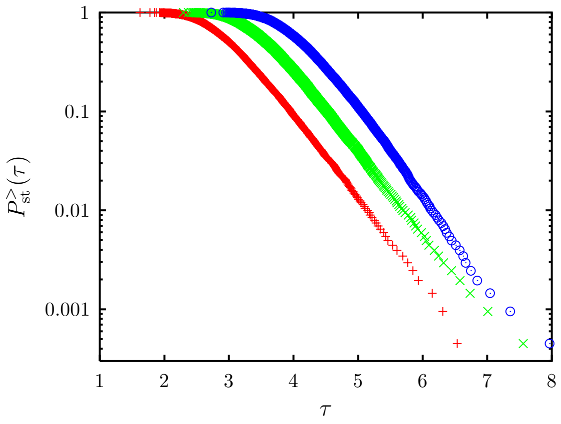

Fig. 1. Probability of reaching the stationary state in time larger than τ, for case I, $q = 2$, $N = 2000$. The values of initial fraction p of opinions +1 are 0.1 (+) 0.2 (×) and 0.7 (⊙).

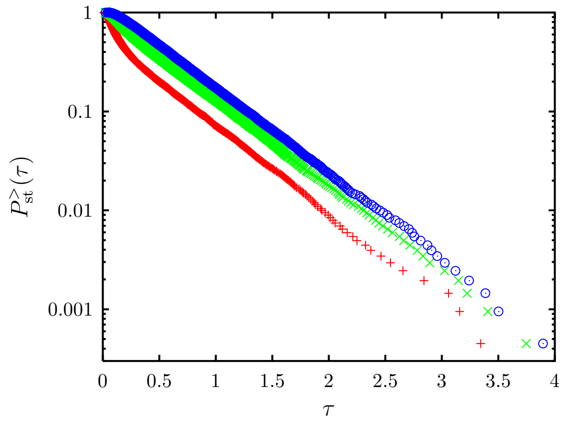

Fig. 2. Probability of reaching the stationary state in time larger than τ, for case II, $q = 2$, $N = 2000$. The values of initial fraction p of opinions +1 are 0.1 (+) 0.2 (×) and 0.7 (⊙).

and by insertion of the solution (34) we deduce that there is homogeneous flow $j = +1$ outward the value $x = -1$, *i.e.* $n = 0$. We may interpret this flow as a consequence of an external source placed somewhere close to the point $x = -1$, *i.e.* $n = 0$. Such a source accounts for the influx of new opinions, or new parties, into the system. It is very reasonable to assume that the source is placed at very small values of n, as new subjects are likely to gain little support initially.

5 Comparison with numerical simulations

We performed numerical simulations of the Sznajd model on fully connected network according to algorithms described in Sections 2.2 (case I) and 2.3 (case II). The main focus was on the dynamical properties, namely the distribution of times needed to reach the homogeneous stationary state. We show in Figures 1 and 2 the probabilities $P^{>}_{\rm st}(\tau)$ that the time $\tau_{\rm st}$ to reach the stationary state is larger that τ. We can clearly see that the probability decays exponentially with τ in both cases I and II.

Let us discuss the case I first. Following the analytical expectation (15) we can fit the exponential tail of the distribution as

$$P^{>}_{\rm st}(\tau) \simeq \exp\left(-\frac{\tau - \langle\tau_{\rm st}\rangle}{\tau_{\rm relax}}\right), \quad \tau \to \infty. \tag{37}$$

The results for $\langle\tau_{\rm st}\rangle$ can be seen in Figure 3, compared with the analytical prediction of equation (12). Similarly in Figure 4 we can compare the fitted relaxation time with the analytical result. Both $\langle\tau_{\rm st}\rangle$ and $\tau_{\rm relax}$ agree satisfactorily with the analytical predictions. The deviations around the value $p = 0.5$ are due to finite size effects; the comparison of the results for system sizes $N = 2000$ and $N = 4000$ supports this interpretation. From equation (12) we can see that $\langle\tau_{\rm st}\rangle$ diverges logarithmically for $N \to \infty$. This is confirmed by the simulation data which fall onto single curve in Figure 3 for different system sizes.

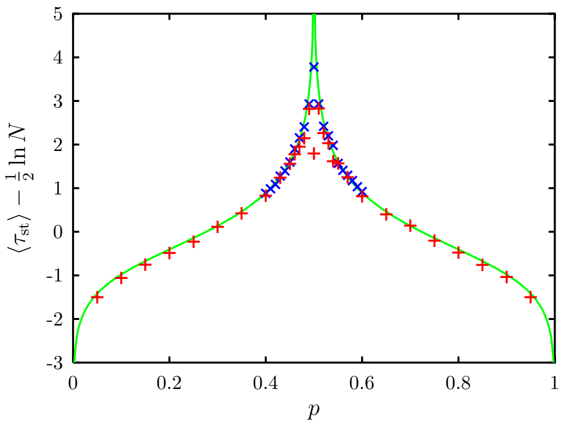

Fig. 3. Average time of reaching the stationary state in dynamics of case I, $q = 2$. The system size is $N = 2000$ (+) and $N = 4000$ (×). The line is the analytic prediction of equation (12)

Now let us turn to the case II. The equation (29) yields the leading term in the tail of the distribution $P^{>}_{\rm st}(\tau)$ and in principle also the corrections to it. As the functions $\phi_c(x)$ are odd for $c = c_l$ with odd l, the next non-zero correction will come from the eigenvalue $c_2 = 12$. Therefore, we expect the behaviour

$$P^{>}_{\rm st}(\tau) \simeq \exp\left(-\frac{\tau - \tau_0}{\tau_{\rm r0}}\right) + a_1 \exp(-\frac{\tau}{\tau_{\rm r1}}),\ \tau \to \infty \tag{38}$$

with

$$\tau_{\rm r0} = \frac{1}{2}, \quad \tau_{\rm r1} = \frac{1}{12}. \tag{39}$$

As in the initial condition $P_0(x) = \delta(x - x_0)$ we have $x_0 = 2p - 1$, we can deduce from equation (30) the following estimate

$$\tau_0 \simeq \ln\sqrt{6\,p\,(1-p)}. \tag{40}$$

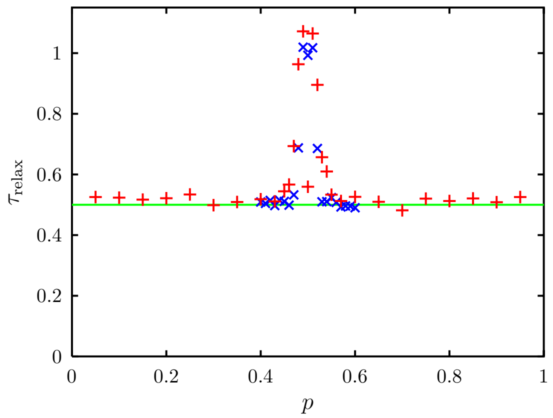

Fig. 4. Relaxation time toward the stationary state in dynamics of case I, $q = 2$. The system size is $N = 2000$ (+) and $N = 4000$ (×). The horizontal line is the analytic prediction of equation (16).

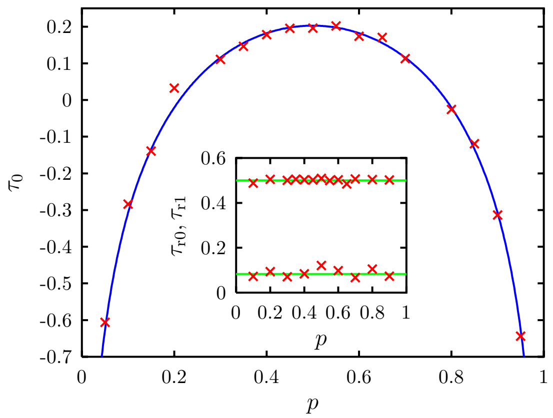

Fig. 5. The fitted parameter τ_0 for reaching the stationary state in dynamics of case II, $q = 2$. The system size is $N = 2000$. The line represents the formula (40). In the inset, the fitted first two relaxation times τ_{r0} and τ_{r1} are shown. The horizontal lines are corresponding analytical predictions from equation (39).

We can see from Figure 5 that it corresponds well to the numerical data. In the inset of Figure 5 we can also see the fitted relaxation times τ_{r0} and τ_{r1}. Also here the correspondence with analytical prediction (39) is good.

6 Conclusions

We formulated a mean-field version of the Sznajd model of opinion formation by putting it on a complete graph. Solving the underlying diffusion equations we found analytical results for several dynamical properties, as well as exact long-time asymptotics. The results differ substantially in the two cases studied: first, the original Sznajd model, where a cluster of identical opinions is necessary to persuade others to join them, and second, the Ochrombel simplification, where also isolated agent can persuade others. Dynamical phase transition was found analytically in the original Sznajd model, while in the Ochrombel version it is absent. This finding agrees with previous numerical results.

The approach to stationary state was the main concern of our calculations. We found that the distribution of times to reach the stationary state has an exponential tail which we were able to calculate analytically. In the case of Ochrombel simplification, we obtained also the corrections and a formula which gives in principle the whole distribution. We compared the analytical results for the tail (and in the Ochrombel case also for the first correction) with numerical simulations and we found good agreement. The method of adjoint equation enabled us to find analytically the average time to reach the stationary state, in both cases.

We found also another signature of the phase transition in the original Sznajd model, expressed by the divergence of the average time to reach the stationary state. Contrary to the Ochrombel case, in the original Sznajd model the average time needed for reaching the stationary state blows up logarithmically with increasing system size. This finding was also confirmed in our numerical simulations.

The analytical treatment provided an explanation of the $1/n$ distribution of votes, documented in Brazilian elections. We found that this behaviour corresponds to long-lived transient state of the dynamics of the Sznajd model with large number of possible opinions, or alternatively to the dynamics of an open version of the Sznajd model, where new opinions may continuously emerge.

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Appendix A: Sznajd model on an arbitrary social network

Our system is composed of N agents placed on nodes of a social network, represented by the graph $\Lambda = (\Gamma, E)$ where Γ is the set of nodes and E set of edges, *i.e.* unordered pairs of nodes. For a node $i \in \Gamma$ we denote $\Gamma_i = \{j \in \Gamma | (i,j) \in E\}$ the set of neighbours of i.

The opinion of the agent i s denoted σ_i. The state of the system is described by the set of opinions of all the agents, $\Sigma = [\sigma_1, \sigma_2, ..., \sigma_N] \in S^{\Gamma}$. The variable $\Sigma(t)$ performs a discrete-time Markov process, defined as follows.

In the case I we iterate the following three steps. First, choose $i \in \Gamma$ at random. Then, choose $j \in \Gamma_i$ randomly among neighbours of i. If $\sigma_i(t) \neq \sigma_j(t)$, nothing happens. However, if $\sigma_i(t) = \sigma_j(t)$, we will choose randomly one of the common neighbours $k \in \Gamma_i \cap \Gamma_j \setminus \{i,j\}$ and set $\sigma_k(t+1) = \sigma_i(t)$.

In the case II we choose $i \in \Gamma$ at random. Then, choose $j \in \Gamma_i$ randomly among neighbours and set $\sigma_j(t+1) = \sigma_i(t)$.

If the graph is random and densely connected, we may approximate it by the complete graph with N nodes, *i.e.* for each pair of nodes $i, j \in \Gamma$ there is an edge connecting them, $(i, j) \in E$. It means that the set of neighbours of a node $i \in \Gamma$ is $\Gamma_i = \Gamma \setminus \{i\}$. This is a kind of a mean-field approximation.

Appendix B: Finding the eigenvectors

We can look for the solution of the equation (22) in the form of power series

$$\phi_c(x) = \sum_{l=0}^{\infty} b_l \, x^l \tag{B.1}$$

and find the recurrence relation for the coefficients

$$b_{l+2} = \left(1 - \frac{c}{(l+1)(l+2)}\right) b_l. \tag{B.2}$$

We should distinguish two cases. Either the sequence of coefficients b_l contains non-zero values for arbitrarily large l, or it is truncated at some order and (B.1) becomes a polynomial. In the former case the solution behaves as $\phi_c(x) \sim (1-x^2)^{-1}$ at $x \to \pm 1$ and must be excluded. The latter case is possible only if

$$c = c_l \equiv (l+1)(l+2) \tag{B.3}$$

for some $l \geq 0$. Moreover, in order to have a solution in the form of a polynomial, we require that $b_1 = 0$ if l in the equation (B.3) is even, and $b_0 = 0$ if l in the equation (B.3) is odd. The following table lists the solution for several lowest eigenvalues (taking $b_0 = 1$ for even l and $b_1 = 1$ for odd l).

l	c_l	$\phi_c(x)$
0	2	1
1	6	x
2	12	$1 - 5x^2$
3	20	$x - \frac{7}{3}x^3$
4	30	$1 - 14x^2 + 21x^4$
$\vdots$	$\vdots$	$\vdots$

(B.4)

In fact, up to a multiplicative constant, the functions $\phi_c(x)$ are Gegenbauer polynomials [53,54].

The same procedure can be used for finding the eigenvectors of the adjoint operator, solving equation (25). We expand the function $\psi_c(x)$ in power series

$$\psi_c(x) = \sum_{l=0}^{\infty} d_l \, x^l \tag{B.5}$$

and find the recurrence relation for the coefficients

$$d_{l+2} = \frac{(l-1)l - c}{(l+1)(l+2)} \, d_l. \tag{B.6}$$

Again we conclude that the only acceptable values of c are given by condition $c = c_l \equiv (l+1)(l+2)$ for $l = 0, 1, 2, ...$ and in this case the eigenvectors are polynomials of order $l+2$ in the variable x. The following table lists the solution for lowest eigenvalues (taking $d_0 = 1$ for even l and $d_1 = 1$ for odd l).

l	c_l	$\psi_c(x)$
0	2	$1 - x^2$
1	6	$x - x^3$
2	12	$1 - 6x^2 + 5x^4$
3	20	$x - \frac{10}{3}x^3 + \frac{7}{3}x^5$
4	30	$1 - 15x^2 + 35x^4 - 21x^6$
$\vdots$	$\vdots$	$\vdots$

(B.7)

It is important to note that the set of right eigenvalues coincides with the set of left eigenvalues, which proves consistency of our approach.

Note that neither $\phi_c(x)$ nor $\psi_c(x)$ are orthogonal polynomials. Instead, they are mutually orthogonal, *i.e.* $\int_{-1}^{1} \phi_c(x)\psi_{c'}(x) \mathrm{d}x = 0$ for $c \neq c'$. This is due to the fact that the operator $\mathcal{L}$ is not self-adjoint.

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